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The Rovella attractor is a homoclinic class

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Abstract. Rovella proved the existence of measure-persistent attractors for flows exhibiting a unique singularity with three real eigenvalues satisfying $\lambda_2 < \lambda_3 < 0 < \lambda_1 < -\lambda_3$ ([Ro]). In this paper we prove that *most* of them are in fact homoclinic classes.

Keywords: Rovella Attractor, Contracting Lorenz Attractor, Homoclinic Class.

Mathematical subject classification: Primary: 37D45; Secondary: 37C10.

1 Introduction

Let X^t be a C^1 flow on a manifold. A compact invariant set of X^t is an *attractor* if it is transitive and maximal invariant in a positively invariant neighborhood of it. A *homoclinic class* of X^t is the closure of the transverse homoclinic orbits associated to a hyperbolic periodic orbit of X^t. One can easily find examples of attractors which are not homoclinic classes as, for instance, the ambient manifold of a minimal flow. Examples which are homoclinic classes are the non-trivial hyperbolic, geometric Lorenz and Henon-like attractors ([KH], [B], [C]). The last two examples are not hyperbolic. In general it is known that a non-trivial attractor of a C^1 generic flow is a homoclinic class.

In this paper we provide more examples of non-hyperbolic attractors which are homoclinic classes. Precisely we shall consider the attractors found by Rovella in his thesis [Ro]. These attractors are measure-persistent and exhibit a unique singularity with real eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ satisfying $\lambda_2 < \lambda_3 < 0 < \lambda_1 < -\lambda_3$. By this reason we shall call them *Rovella attractors* although some authors use the term contracting Lorenz attractor in opposite to the classical geometric Lorenz attractor which satisfies the eigenvalue relation $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$. It

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turns out that the Rovella attractors are neither hyperbolic (since they display regular and singular orbits in the same transitive set) nor singular-hyperbolic (since they are transitive and display non-Lorenz-like singularities [BDV]). In this paper we prove however that most Rovella attractors are homoclinic classes.

Let us state our result in a precise way. An *attracting set* of X^t is a compact invariant set Λ for which there is a neighborhood U such that

$$\Lambda = \bigcap_{t \geq 0} X^t(U).$$

The set U above can be chosen positively invariant, i.e. $X^t(U) \subset U$. Hereafter we shall call such a neighborhood *isolating block*. An isolating block can be chosen arbitrarily close to Λ as well. If U is an isolating block and Y^t is a flow close to X^t then the set

$$\Lambda_Y = \bigcap_{t \geq 0} Y^t(U)$$

is an attracting set of Y^t. This attractor is often called the continuation of Λ. An invariant set is *transitive* if it is $\omega(q)$ for some q on it. Recall that $\omega(q)$, the *omega-limit set of* q, is the accumulation point set of the positive orbit of q under X^t. An *attractor* is a transitive attracting set.

Given a subset S in a Banach E we say that $x \in S$ is a *point of k-dimensional full density* of S if there is a codimension k submanifold $N \subset E$ containing x such that if M is a k-dimensional submanifold of E intersecting S transversally, then every point $y \in N \cap M$ satisfies

$$\lim_{r \to 0^+} \frac{m(B_r(y) \cap S)}{m(B_r(y))} = 1,$$

where m is the Lebesgue measure in M and $B_r(y)$ is the r-ball centered at y in M.

We say that an attractor Λ of X is *persistent in an almost k-persistent way* if there is an isolating block U of Λ such that X is a k-dimensional full density point of

$$S = \{Y : Y \text{ is close to } X \text{ and } \Lambda_Y \text{ is an attractor of } Y\}.$$

In his thesis A. Rovella proved the following result (see part (b) of the Theorem in [Ro] p. 235).

Theorem 1.1. *There is a C^∞ vector field X_0 in $\mathbb{R}^3$ having an attractor Λ containing a singularity with eigenvalues satisfying $\lambda_2 < \lambda_3 < 0 < \lambda_1 < -\lambda_3$ such that Λ is* **persistent** *in an almost 2-persistent way.*

Motivated by the above definitions and result we introduce the following definition: We say that an attractor Λ of X is a *homoclinic class in an almost k-persistent way* if there is an isolating block U of Λ such that X is a k-dimensional full density point of the set

$$S = \{Y : Y \text{ is close to } X \text{ and } \Lambda_Y \text{ is a homoclinic class of } Y\}.$$

In this paper we improve Theorem 1.1 in the following way.

Theorem 1.2. *There is a C^∞ vector field X_0 in $\mathbb{R}^3$ having an attractor Λ containing a singularity with eigenvalues satisfying $\lambda_2 < \lambda_3 < 0 < \lambda_1 < -\lambda_3$ such that Λ is a* **homoclinic class** *in an almost 2-persistent way.*

Although the unperturbed vector field X_0 and its corresponding attractor Λ in Theorem 1.2 are exactly the ones in Theorem 1.1 the attractors obtained in our theorem are not so. Actually, to prove our theorem, we shall prove that the set of vector fields for which the attractor in Theorem 1.1 is a homoclinic class is large enough to obtain homoclinic classes in an almost 2-persistent way. Observe that Theorem 1.2 implies Theorem 1.1 by the Birkhoff-Smale Theorem [KH].

This paper is organized as follows. In Section 2 we introduce the Rovella attractor and in Section 3 we prove that the corresponding one-dimensional maps are LEO (locally eventually onto). In Section 4 we prove Theorem 1.2.

2 Construction of X_0 and Λ

We just recall Section 1 p. 237 in [Ro].

Start with a C^∞ vector field X_0 in $\mathbb{R}^3$ such that $O = (0, 0, 0)$ is a singularity. The eigenvalues of O are real numbers $\lambda_1, \lambda_2, \lambda_3$ satisfying $\lambda_2 < \lambda_3 < 0 < \lambda_1 < -\lambda_3$. The corresponding eigenspaces will be the coordinate axis. We will also assume that X_0 is linear in the cube $\{(x, y, z) : |x|, |y|, |z| \leq 1\}$. Both trajectories of the unstable manifold of O intersect the top rectangle Q of the cube.

This rectangle is divided by the stable manifold of 0 in two subrectangles the union of which is denoted by Q^*. There are two return maps Π_{loc}, Π_{far} induced by the flow from Q^* to $\{x = \pm 1\}$ and from $\{x = \pm 1\}$ back to Q. The composition $\Pi_0 = \Pi_{far} \circ \Pi_{loc}$ is the return map associated to Q and its image $\Pi_0(Q^*)$ consists of two cusp triangles as in Figure 1-(a). We also assume that Π has the form

$$\Pi_0(x, y) = (f_0(x), g_0(x, y))$$

so Π_0 preserves the constant vertical foliation $\{x = cnt\}$ in Q. We assume that this foliation is contracted by Π_0. We further assume the following hypotheses:

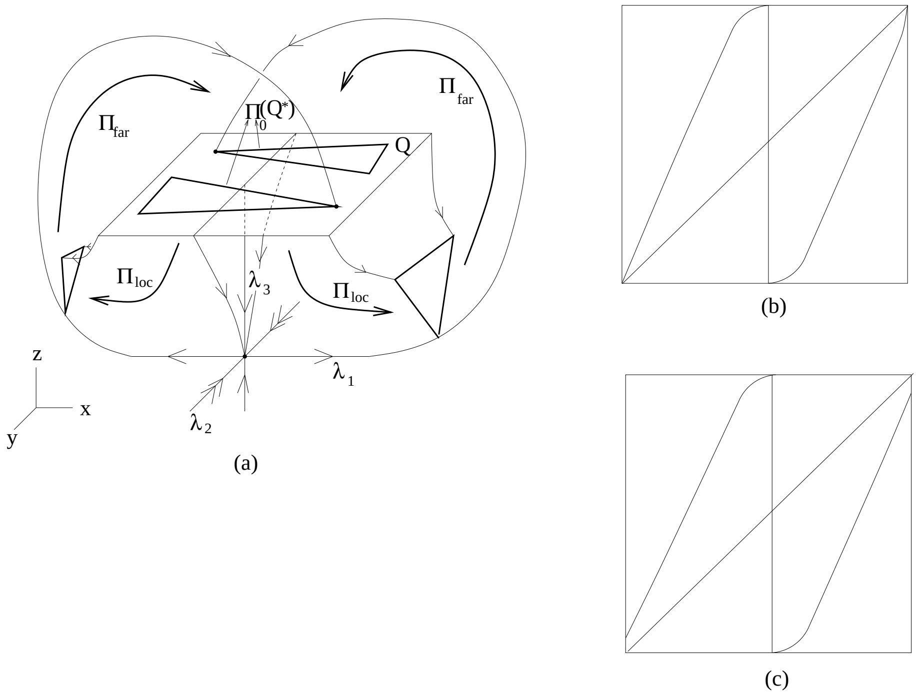

Figure 1:

(H1) The order of f_0' at $x = 0$ is $s - 1$ where $s > 1$ is a fixed constant.

(H2) f_0 has a discontinuity at $x = 0$ with $f_0(0+) = -1$, $f_0(0-) = 1$.

(H3) $f_0'(x) > 0$ for $x \neq 0$.

(H4) $\max_{x>0} f_0'(x) = f_0'(1)$ and $\max_{x<0} f_0'(x) = f_0'(-1)$.

(H5) 1 and -1 are preperiodic repelling, that is, there are positive integers k^-, k^+, n^-, n^+ such that

$$f_0^{k^+ + n^+}(1) = f_0^{k^+}(1), \quad (f_0^{n^+})'(f_0^{k^+}(1)) > 1$$

and

$$f_0^{k^- + n^-}(-1) = f_0^{k^-}(-1), \quad (f_0^{n^-})'(f_0^{k^-}(-1)) > 1.$$

(H6) f_0 has negative schwarzian derivative.

The construction implies that there is a compact positively invariant neighborhood U of the cube above. Define

$$\Lambda = \bigcap_{t \geq 0} X_0^t(U).$$

This ends the construction of X_0 and Λ.

3 Proofs

In this section we prove that the attractor Λ previously defined is a homoclinic class in a 2-parameter almost persistent way. By definition we need to prove that X_0 is a 2-dimensional full density point of

$$S = \{Y : Y \text{ is close to } X \text{ and } \Lambda_Y \text{ is a homoclinic class of } Y\}.$$

For this we need to define a codimension two submanifold N. By the Proposition in [Ro] p. 241 we have that for every X in a neighborhood $\mathcal{U}$ of X_0 there is a one-dimensional foliation in the isolating block U of Λ which is stable and varies continuously with X. With this we can define a one-dimensional map f_X which is the continuation of the map f_0 in the previous section. As in [Ro] p. 246 we define N as the set of $X \in \mathcal{U}$ such that

$$f_X^{k^+}(1) \quad \text{and} \quad f_X^{k^-}(-1)$$

are preperiodic of periods n^+ and n^-.

Now, let M be a C^3 2-dimensional submanifold of $\mathcal{U}$ intersecting N transversally. To prove the limit in the definition of a k-dimensional full density point, we only need to consider, as in [Ro] p. 247, a one-parameter family $\{Y_a\}_{a \geq 0}$ in M such that the maps $a \to f_{Y_a}(\pm 1)$ has derivative 1 at 0. We will prove that $a = 0$ is a full density point of the set of parameters for which Λ_{Y_a} is a homoclinic class of Y_a. According to the arguments in [Ro] p. 247 this suffices. Previously we shall prove that the associated family $f_a = f_{Y_a}$ of one-dimensional maps satisfy the following theorem.

Theorem 3.1. *There is a positive Lebesgue measure subset E of the parameter space such that*

1. $\lim_{a \to 0} \frac{m(E \cap [0,a))}{a} = 1.$
2. *If $a \in E$, then f_a is LEO.*

We will use three properties of the one-dimensional Lorenz-like maps studied by [Ro]. More precisely, let $I \subset [-1, 1]$ be a compact interval and $f : I \to I$ be a map such that $f(I) \subset I$ with a discontinuity at the origin. Set $c_k^{\pm} = \lim_{x \to 0^{\pm}} f^k(x)$ for $k \geq 0$, so the properties can be stated as follows:

A0) Outside the origin f is of class C^3 and with negative Schwarzian derivative, and also satisfies

$$K_2|x|^{s-1} \leq f^{'}(x) \leq K_1|x|^{s-1}.$$

For some constants K_1, K_2 and s with $s > 1$.

A1) $(f^n)^{'}(c_1^{\pm}) > \lambda_c^n$, for some $\lambda_c > 1$, and for $n \geq 1$.

A2) $|f^{n-1}(c_1^{\pm})| > e^{-\alpha n}$ some α small enough, and all $n \geq 1$.

In [Ro], section IV, it is proved that for the associated one-parameter family of maps $\{f_a\}_{a \in [0,2)}$ obtained as specified at the beginning of this section there is a positive Lebesgue measure subset $E \subset [0, 2)$ with $0 \in E$ as a Lebesgue full density point such that the map

$$f = f_a, \quad \forall a \in E$$

satisfies A0-A2. So, we only need to prove that if f_a satisfies A0-A2 then it is LEO, redefining the set E to $E \cap [0, r)$ for small enough r if necessary.

The basic strategy is to reduce the non-uniform hyperbolicity of the dynamics of our maps to that of piecewise uniformly expanding maps. That is what conditions A1-A2 are for, which express a kind of expansiveness. Our proof follows closely the arguments in section 4 of [M] which in turns is based on the arguments by L.-S. Young in [Yo] for unimodal maps.

Before the proof we need the following lemmas the first of which corresponds to (P1) in [Yo].

Lemma 3.2. *There exists $r > 0$ such that for every $a \in [0, r]$ the map $f = f_a$ satisfies the following property: There are constants $\sigma_0 > 1$, $b > 0$ and $\delta_0 > 0$ such that for any $0 < \delta \leq \delta_0$ there is $c(\delta) > 0$ such that, given any $x \in I$ and $n \geq 1$*

1) *If $x, f(x), \ldots, f^{n-1}(x) \notin (-\delta, \delta)$ then $(f^n)^{'}(x) \geq c(\delta)\sigma_0^n$.*

2) *If in addition, $f^n(x) \in (-\delta, \delta)$ then $(f^n)^{'}(x) \geq b\sigma_0^n$.*

Proof. This was proved in other form by Rovella in [Ro], see lemmas 1, 1.1, 1.2 and their proofs, in the mentioned article. □

Now, let $I_m = (e^{-m-1}, e^m)$ for $m > 0$, let $I_m = -I_{-m}$ for $m < 0$, and $I_m^+ = I_{m-1} \cup I_m \cup I_{m+1}$, $\delta = e^{-\Delta}$, with $\Delta \in \mathbb{N}$ and choose β such that $\frac{s}{s-1}\alpha > \beta > \frac{s+1}{s}\alpha$, where s is the fixed constant in (H1).

Let $p(m)$ be the largest integer p such that for every $x \in I_m^+$ and $j = 1, \dots, p$:

$$\begin{aligned} |f^j(x) - f^{j-1}(c_1^+)| &= |f^j(x) - c_j^+| \le e^{-\beta j} \quad \text{if} \quad m > 0 \quad \text{and} \\ |f^j(x) - f^{j-1}(c_1^-)| &= |f^j(x) - c_j^-| \le e^{-\beta j} \quad \text{if} \quad m < 0\,. \end{aligned}$$

The time interval $1, \dots, p(m)$ is called the bound period for I_m^+.

The following lemma corresponds to (P2) in [Yo], Lemma 2.2 in [BC2] and Lemma 1 and 2 in [BC1] for quadratic maps. Indeed, you can find part (b) inside the proofs of the latter mentioned lemmas. The main difference is the point of discontinuity and also that we are not dealing with exactly a “quadratic” map but with some maps that “looks like” a quadratic map in the sense of Property A0.

The proof is essentially contained in [M] and we included it here for completeness.

Lemma 3.3. *For each* $|m| > \Delta$, $p(m)$ *has the following properties.*

a) *There is a constant* $C(\alpha, \beta)$ *such that:*

i)

$$\frac{1}{C} \le \frac{(f^j)'(y)}{(f^j)'(c_1^+)} \le C \qquad \text{if } y \in [-1, f(e^{-|m|+1})]\,,$$

ii)

$$\frac{1}{C} \le \frac{(f^j)'(y)}{(f^j)'(c_1^-)} \le C \qquad \text{if } y \in [f(-e^{-|m|+1}), 1]\,.$$

for $j = 0, \dots, p(m)$.

b)

$$\frac{s|m|}{\beta + \log 4} - K \le p(m) \le \frac{s+1}{\beta + \log \lambda_c}|m|$$

$$\textit{where} \;\; K = \frac{\beta + \log(K_1/s) + s}{\beta + \log 4}.$$

c) *If* $z \in I_m^+$ *then*

$$(f^p)'(f(z)) \geq \frac{1}{C}\lambda_c^p$$

and

$$(f^{p+1})'(z) \geq \frac{1}{C}(\lambda_c^{1/s})^p M\,,$$

where $M = e^{-\alpha}(s/(CK_1))^{(s-1)/s} \cdot K_2$ *and* $p = p(m)$.

d)
$$(f^{p+1})'(x) \geq \exp\left(\left(1 - \beta\frac{s+2}{\beta + C}\right)|m|\right)$$

where $p = p(m)$ *and for* $x \in I_m^+$.

Proof. Suppose $y \in [c_1^+,\ f(e^{-|m|+1})]$ (for $y \in [f(e^{-|m|+1}), c_1^-]$ the proof is similar).

First of all note that

$$\frac{(f^k)'(f(z))}{(f^k)'(c_1^-)} = \prod_{j=1}^{k}\frac{f'(f^j(z))}{f'(c_j^-)} = \prod_{j=1}^{k}\left(1 + \frac{f'(f^j(z)) - f'(c_j^-)}{f'(c_j^-)}\right)$$

so we only have to get a uniform bound for

$$\sum_{j=1}^{k}\left|\frac{f'(f^j(z)) - f'(c_j^-)}{f'(c_j^-)}\right|.$$

Now, f has negative Schwarzian derivative in B_j^- since $0 \notin B_j^- = [c_j^- - e^{-\beta j}, c_j^- + e^{-\beta j}]$, and as long as $f^j(z) \in B_j^-$ we have that

$$\left|\frac{f'(f^j(z)) - f'(c_j^-)}{f'(c_j^-)}\right| \leq |f''(y)|\left|\frac{f^j(z) - c_j^-}{f'(c_j^-)}\right| \leq A|y|^{s-2}\left|\frac{f^j(z) - c_j^-}{f'(c_j^-)}\right|.$$

Then from condition A0 we obtain:

$$\sum_{j=1}^{k}\left|\frac{f'(f^j(z)) - f'(c_j^-)}{f'(c_j^-)}\right| \leq \frac{A}{K_2}\sum_{j=1}^{k}\frac{e^{-\beta j}}{e^{-\alpha j}}\,.$$

The right side is bounded because $\beta > \alpha$ from the condition impose onβ immediately after the proof of Lemma 3.2. Now part (a) follows making $y = f(z)$ with $z \in (0, e^{-|m|+1})$.

To prove (b).

For $x \in I_m^+$ we have, assuming $m \geq 0$ to fix ideas,

$$e^{-\beta p} \geq |f^p(x) - c_p^+| = |f^{p-1}(f(x)) - f^{p-1}(c_1^+)| = (f^{p-1})'(y)|f(x) - (c_1^+)|$$

for some $y \in [c_1^+, f(x)] \subset [-1, f(e^{-|m|+1})]$ so,

$$\begin{aligned} |f^p(x) - f^{p-1}(c_1^+)| &= (f^{p-1})'(y)|f(x) - (c_1^+)| \geq (f^{p-1})'(y)K_2\frac{|x|^s}{s} \\ &\geq \frac{(f^{p-1})'(c_1^+)}{C_1}\frac{K_2}{s}e^{(-|m|-2)s} \\ e^{-\beta p} &\geq \frac{\lambda_c^{(p-1)}}{C_1}\frac{K_2}{s}e^{-|m|s}e^{-2s}. \end{aligned}$$

So we have the following bound for p,

$$\log\left(\frac{K_2}{C_1 s}\right) - |m|s - 2s + \log\lambda_c p - \log\lambda_c \leq -\beta p$$

that is,

$$p \leq \frac{s|m|}{\log\lambda_c + \beta} + \frac{\log\lambda_c + 2s - \log\frac{K_2}{C_1 s}}{\log\lambda_c + \beta}.$$

If $|m|$ is large enough we can write,

$$p \leq \frac{(s+1)|m|}{\log\lambda_c + \beta}.$$

For the other inequality, from the definition of p, there must exists a $z \in I_m^+$ such that

$$e^{-\beta(p+1)} \leq \left|f^p(f(z)) - f^p(c_1^+)\right| \leq (f^p)'(y)\left|f(z) - (c_1^+)\right|.$$

Supposing that $f' \leq 4$, we obtain,

$$e^{-\beta(p+1)} \leq 4^p K_1\frac{z^s}{s} \leq 4^p\frac{K_1}{s}e^{(-|m|+1)s}$$

so

$$-\beta(p+1) \leq p\log 4 + \log(K_1/s) + (-|m|+1)s$$

which implies that

$$p \geq \frac{s|m|}{\beta + \log 4} - \frac{\log(K_1/s) + s + \beta}{\beta + \log 4}.$$

Now, to prove part (c), first observe that the first claim in (c) is a direct consequence of part (a) and A1. The second one can be obtained as follows. Let $z \in I_m^+$ and $p = p(m)$, then

$$
\begin{aligned}
(f^{p+1})'(z) = (f^p)'(f(z)).\,f'(z) &\geq K_2|z|^{s-1}(f^p)'(f(z)) \\
&\geq \frac{K_2}{C}|z|^{s-1}(f^p)'(c_1^-)\,.
\end{aligned}
\tag{1}
$$

We can estimate the value of $|z|$ from the inequality

$$
\begin{aligned}
e^{-\beta(p+1)} \leq \left| f^{p+1}(z) - c_{p+1}^- \right| &= (f^p)'(\xi) \left| f(z) - c_1^- \right| \\
&\leq K_1 C (f^p)'(c_1^-)\frac{|z|^s}{s}
\end{aligned}
\tag{2}
$$

for some $\xi \in (f(z), c_1^-)$ from the Mean Value Theorem. For this ξ there exists y satisfying the conditions in part (a) and such that $f(y) = \xi$. The last inequality is due to A0. So the inequality above is a consequence of the Mean Value Theorem and part (a).

Rewriting the equation, it stands that:

$$
|z|^s \geq \frac{s}{CK_1}\left|(f^p)'(c_1^-)\right|^{-1} e^{-\beta(p+1)}.
$$

Combining this last inequality with (1) we obtain

$$
(f^{p+1})'(z) \geq \frac{K_2}{C}\left(\frac{e^{-\beta}s}{CK_1}\right)^{\frac{s-1}{s}} (\lambda_c^{1/s})^k \cdot e^{-\beta k(\frac{s-1}{s})}\,.
$$

Since $\beta < \frac{s}{s-1}\alpha$ we have

$$
(f^{p+1})'(z) \geq \frac{K_2}{C}\left(\frac{e^{-\beta}s}{CK_1}\right)^{\frac{s-1}{s}} \left(\lambda_c^{1/s}e^{-\alpha}\right)^p\,,
$$

leading to

$$
(f^{p+1})'(z) \geq \frac{1}{C}\left(\lambda_c^{1/s}e^{-\alpha}\right)^p M\,,
$$

where $M = e^{-\alpha}(s/(CK_1))^{(s-1)/s} \cdot K_2$. So part (c) is proved.

This ends the proof of Lemma 3.3, because part (d) is an easy consequence of the second assertion in part (c). □

Proof of Theorem 3.1. As we said before, it is enough to proof that for a in a small neighborhood of 0 conditions A0, A1 and A2 implies that f_a is LEO. The proof is based on an argument in [Yo].

In [Yo] it was used the fact that the the initial map f_0 has a fixed point with dense backward orbit in $[-1, 1]$. Here we don't have such fixed point for f_0 but we can construct f_0 in the following way. First we consider a map F with all the properties of f_0 *except that it fixes both* 1 *and* -1 (note however that F does not come from the return map of an attractor). We can choose F conjugated to $1-2x \pmod{\mathbb{Z}}$ so it has a periodic point $z \in (-1, 1)$ whose unstable manifold is the whole $[-1, 1]$. Since $z \in (-1, 1)$ the conjugation implies that the backward orbit of z under F is dense in $[-1, 1]$. In particular, z has F-preimages in both I_Δ and $I_{-\Delta}$. Afterward we obtain f_0 by perturbing F in a way that $f_0(-1) > -1$ and $f_0(1) < 1$. We choose f_0 close to F enough such that the f_0-continuation z_0 of z still has f_0-preimages in both I_Δ and $I_{-\Delta}$. This finish the construction of f_0.

As to be in I_Δ and $I_{-\Delta}$ is an open condition in the parameter space we have that the f_a-continuation z_a of z_0 still has preimages in both I_Δ and $I_{-\Delta}$ for all $a > 0$ close to 0. So, the conclusion remains valid only reducing E to $E \cap [0, r)$ for small enough $r > 0$.

Next we follow [Yo], pag. 127. Let $f = f_a, a \in E$. First we prove that for all $I \subset [-1, 1]$, there exists $n_0 = n_0(I)$ such that $f^{n_0}(I) \supset I_\Delta$ or $I_{-\Delta}$. According to Lemma 3.2, if the iterates of I do not intersect $(-\delta, \delta)$ the length of the iterates increases, so there is some $f^j(I)$ that intersects $(-\delta, \delta)$. If $f^j(I)$ does not contain some I_k, keep iterating, and note that using Lemma 3.3 we have $|f^p(f^j(I))| \gg |f^j(I)|$, $p = p(x)$ for $x \in f^j I$. After finitely many returns to $(-\delta, \delta)$, there must exist j_1 and $k_1 \in \mathbb{Z}^+$ such that $f^{j_1}(I) \supset I_{k_1}$ or I_{-k_1}. Consider $f^j(I_{k_1})$, $j = 1, 2, \ldots$, and let j_2 be the first time (after the bound period of some $x \in I_{k_1}$) such that $f^j(I_{k_1}) \supset$ some I_k. Since $|f^{j_2}(I_{k_1})| \gg |I_{k_1}|$, $f^{j_2}(I_{k_1})$ must contain some I_{k_2} or I_{-k_2} with $0 < k_2 < k_1$. We then consider $f^j(I_{k_2})$ and repeat the argument until some $f^j(I_{k_n}) \supset I_\Delta$ or $I_{-\Delta}$.

To finish, since there is an $n_1 \in \mathbb{Z}^+$ such that $z_a \in f^{n_1}(I_\Delta)$ or $f^{n_1}(I_{-\Delta})$, where z_a is the aforementioned continuation of z_0. Observe that for any $f = f_a$, $a \in E$, and $\hat{I}$ some interval containing z_a there exists $n_2 = n_2(\hat{I})$ such that $f^{n_2}(\hat{I}) \supset [f(0^+), f(0^-)]$, i.e. f is LEO.

So Theorem 3.1 follows. □

4 Proof of Theorem 1.2

Theorem 3.1 in the previous section implies that the vector field associated to the Rovella attractor originates a LEO one dimensional map in an almost 2-persistent way.

That is, in order to prove Theorem 1.2 it is left to prove that a vector field with a periodic orbit that originates a LEO one dimensional map is a homoclinic class. For this we use the argument in [B].

Consider the two-dimensional map $\Pi : Q^* \to Q^*$, the return map associated to the Rovella attractor. Define

$$A_\Pi^\infty = \bigcap_{n=1}^{\infty} \overline{A_\Pi^n}$$

the attractor for this return map, where

$$A_\Pi^n = \{x = \Pi^n(z) : z, \Pi(z), \ldots, \Pi^{n-1}(z) \notin \Gamma\},$$

and Γ is the intersection of Q with the local stable manifold of the singularity.

It suffices to prove that A_Π^∞ is a homoclinic class of Π.

Now, since there exists a periodic point p of period two for the one dimensional map associated to this return map, so there is a periodic point p of period two for the return map (recall that we have a contracting foliation for Π).

So we are going to prove that the homoclinic class of p, $H_\Pi(p)$, is A_Π^∞. Obviously $H_\Pi(p) \subset A_\Pi^\infty$ so we only need to prove $A_\Pi^\infty \subset H_\Pi(p)$.

Take $y \in A_\Pi^\infty$ and $\epsilon > 0$, we need to prove $H_\Pi(p) \cap B_\epsilon(y) \neq \emptyset$. Take n large enough so that if $z \in Q$ satisfies that $\Pi^i(z)$ is defined for all $0 \leq i \leq n-1$ and $\Pi^n(z) \in B_{\epsilon/2}(y)$, then Π^n carries the component of the stable manifold $W^s(z)$ containing z inside $B_\epsilon(y)$, i.e., $\Pi^n(W^s(z)) \subset B_\epsilon(y)$. For such an n we have $y \in \overline{A_\Pi^n}$ by definition, so there is $x_n \in B_\epsilon(y) \cap A_\Pi^n$. Again by definition there is z_n such that $x_n = \Pi^n(z_n)$ with $z_n, \Pi(z_n), \ldots, \Pi^{n-1}(z_n)$ well defined. The LEO property of the one-dimensional map associated to Π implies that $W^s(p)$ is dense in Q and that $W^u(p)$ intersects $W^s(z_n)$. Then, there is $h_n \in H_\Pi(p)$ arbitrarily close to $W^s(z_n)$. In particular, we have $\Pi^i(h_n)$ stays close to $\Pi^i(W^s(z_n))$ for all $i = 0, \ldots, n$, so $\Pi^n(h_n) \in B_\epsilon(y)$ by the property of n. Since $H_\Pi(p)$ is Π-invariant we get the result. □

References

[B] S. Bautista. *The geometric Lorenz attractor is a homoclinic class*, Bol. Mat. (N.S.) **11** (2004), n. 1, 69–78.

[BC1] M. Benedicks and L. Carleson. *On iterations of* $1 - ax^2$ *on* $(-1, 1)$. Ann. of Math. (2) **122** (1985), n. 1, 1–25.

[BC2] M. Benedicks and L. Carleson. *The dynamics of the Hénon map*. Ann. of Math. (2) **133** (1991), n. 1, 73–169.

[BDV] C. Bonatti, L. Diaz and M. Viana. *Dynamics beyond uniform hyperbolicity. A global geometric and probabilistic perspective*, Encyclopaedia of Mathematical Sciences, 102. Mathematical Physics, III. Springer-Verlag, Berlin, 2005.

[C] Y. Cao. *The transversal homoclinic points are dense in the codimension*-1 *Henon-like strange attractors*, (English. English summary) Proc. Amer. Math. Soc. **127** (1999), n. 6, 1877–1883.

[GW] J. Guckenheimer and R.F. Williams. *Structural stability of Lorenz attractors*, Publ. Math. IHES **50** (1979), 307–320.

[KH] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems.* Encyclopedia of Mathematics and its Applications, 54. Cambridge University Press, Cambridge, 1995.

[Lo] E.N. Lorenz. *Deterministic non periodic flow*, J. Atmosph. Sci. **20** (1963), 130–141.

[M] R. Metzger. *Sinai-Ruelle-Bowen measures for contracting Lorenz maps and flows*, Annales de L'I.H.P. Jour. d'Analyse **17** (2000), n. 2, 247–276.

[Ro] A. Rovella. *The dynamics of perturbations of the contracting Lorenz attractor*, Bull. Braz. Math. Soc. **24** (1993), 233–259.

[Yo] L-S. Young. *Decay of correlations for certain quadratic maps*, Comm. Math. Phys. **146** (1992), 123–138.

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